HDG Methods for Elasticity Problems

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Outline

Goals:

- Linear elasticity
- Nonlinear elasticity
- Elastodynamics

Notation: Div, Curl, Grad

Divergence (**Div**) of a vector:

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

Rotation (**Curl**) of a vector:

$$\nabla \times \boldsymbol{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \; \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \; \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)$$

Gradient (**Grad**) of a vector:

$$\boldsymbol{H} \equiv \nabla \boldsymbol{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

Notation: Strain and Rotation Tensor

Symmetric gradient (strain) tensor:

$$\boldsymbol{\varepsilon} \equiv \frac{\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T}{2} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

Anti-symmetric gradient (rotation) tensor:

$$\boldsymbol{\omega} \equiv \frac{\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^T}{2} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{bmatrix}$$

Note that

$$\boldsymbol{H} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}.$$

Notation: Stress tensor

Stress tensor:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

For linear elasticity, the stress tensor is related to the strain tensor by

$$\boldsymbol{\sigma} = \boldsymbol{\mathcal{C}}\boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} = \boldsymbol{\mathcal{S}}\boldsymbol{\sigma}$$

where $\boldsymbol{\mathcal{C}}$ is the elastic stiffness (fourth-order) tensor and $\boldsymbol{\mathcal{S}}$ is the elastic compliance (fourth-order) tensor. Assuming that they are symmetric.

Linear Elasticity: Stress Formulation

The governing equations are

$$-\nabla \cdot \boldsymbol{\sigma} \;=\; \boldsymbol{b}, \qquad \text{in } \Omega. \tag{1}$$

Here $\boldsymbol{b}$ is a body force

Since $\boldsymbol{\varepsilon} = \boldsymbol{\mathcal{S}}\boldsymbol{\sigma} = \frac{\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T}{2}$, we obtain

$$\begin{aligned} 2\boldsymbol{\mathcal{S}}\boldsymbol{\sigma} - (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) \;&=\; 0, \qquad \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{\sigma} \;&=\; \boldsymbol{b}, \qquad \text{in } \Omega. \end{aligned} \tag{2}$$

This is the **stress formulation**.

Linear Elasticity: Strain Formulation

The governing equations are

$$-\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{b}, \qquad \text{in } \Omega. \tag{3}$$

Here $\boldsymbol{b}$ is a body force

Since $\boldsymbol{\varepsilon} = \frac{\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T}{2}$ and $\boldsymbol{\sigma} = \boldsymbol{\mathcal{C}} \boldsymbol{\varepsilon}$, we obtain

$$\begin{aligned} 2\boldsymbol{\varepsilon} - (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) &= 0, \qquad \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{\mathcal{C}} \boldsymbol{\varepsilon} &= \boldsymbol{b}, \qquad \text{in } \Omega. \end{aligned} \tag{4}$$

This is the **strain formulation**.

Linear Elasticity: Boundary Conditions

Displacement boundary condition:

$$\boldsymbol{u} = \boldsymbol{g}_D, \quad \text{on } \Gamma_D. \tag{5}$$

Stress boundary condition:

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{g}_N, \quad \text{on } \Gamma_N. \tag{6}$$

The complete governing equations:

$$\begin{aligned}
2\boldsymbol{\mathcal{S}}\boldsymbol{\sigma} - (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) &= 0, && \text{in } \Omega, \\
-\nabla \cdot \boldsymbol{\sigma} &= \boldsymbol{b}, && \text{in } \Omega, \\
\boldsymbol{u} &= \boldsymbol{g}_D, && \text{on } \Gamma_D, \\
\boldsymbol{\sigma} \cdot \boldsymbol{n} &= \boldsymbol{g}_N, && \text{on } \Gamma_N.
\end{aligned} \tag{7}$$

Approximation Spaces

We introduce the following spaces

$$W_h^k = \{w \in L^2(\mathcal{T}_h) \; : \; w|_K \in \mathcal{P}_k(K), \forall K \in \mathcal{T}_h\},$$

$$\boldsymbol{V}_h^k = \{\boldsymbol{v} \in [L^2(\mathcal{T}_h)]^d \; : \; \boldsymbol{v}|_K \in [\mathcal{P}_k(K)]^d, \forall K \in \mathcal{T}_h\},$$

$$\boldsymbol{Q}_h^k = \{\boldsymbol{E} \in [L^2(\mathcal{T}_h)]^{d\times d} \; : \; \boldsymbol{E}|_K \in [\mathcal{P}_k(K)]^{d\times d}, \forall K \in \mathcal{T}_h\},$$

$$\boldsymbol{S}_h^k = \{\boldsymbol{\zeta} \in [L^2(\mathcal{T}_h)]^{d\times d} \; : \; \boldsymbol{\zeta}|_K \in [\mathcal{P}_k(K)]^{d\times d}, \boldsymbol{\zeta} = \boldsymbol{\zeta}^T, \forall K \in \mathcal{T}_h\},$$

$$\boldsymbol{M}_h^k = \{\boldsymbol{\mu} \in [L^2(\mathcal{E}_h)]^d \; : \; \boldsymbol{\mu}|_F \in [\mathcal{P}_k(F)]^d, \forall F \in \mathcal{E}_h\}.$$

Note that $\boldsymbol{S}_h^k$ is the space of polynomials for symmetric tensors.

HDG Method for Linear Elasticity

We seek $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) \in \boldsymbol{S}_h^k \times \boldsymbol{V}_h^k \times \boldsymbol{M}_h^k$ such that

$$\begin{aligned}
(\boldsymbol{\mathcal{S}}\boldsymbol{\sigma}_h, \boldsymbol{\zeta})_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla \cdot \boldsymbol{\zeta})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{\zeta} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\
(\boldsymbol{\sigma}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}, \boldsymbol{w} \rangle_{\partial \mathcal{T}_h} &= (\boldsymbol{b}, \boldsymbol{w})_{\mathcal{T}_h}, \\
\langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} + \langle \widehat{\boldsymbol{u}}_h - \boldsymbol{g}_D, \boldsymbol{\mu} \rangle_{\Gamma_D} + \langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n} - \boldsymbol{g}_N, \boldsymbol{\mu} \rangle_{\Gamma_N} &= 0,
\end{aligned} \tag{8}$$

for all $(\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\mu}) \in \boldsymbol{S}_h^k \times \boldsymbol{V}_h^k \times \boldsymbol{M}_h^k$, where

$$\widehat{\boldsymbol{\sigma}}_h \boldsymbol{n} = \boldsymbol{\sigma}_h \cdot \boldsymbol{n} - \boldsymbol{\tau}(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h). \tag{9}$$

Here $\boldsymbol{\tau}$ is the stabilization tensor.

Nonlinear Elasticity

We consider the static equilibrium for a nonlinear elastic body

$$
\begin{aligned}
\boldsymbol{F} - \nabla \boldsymbol{\varphi} &= 0 && \text{in } \Omega, && \text{(10a)}\\
-\nabla \cdot \boldsymbol{P}(\boldsymbol{F}) &= \boldsymbol{b} && \text{in } \Omega && \text{(10b)}\\
\boldsymbol{\varphi} &= \boldsymbol{g}_D, && \text{on } \Gamma_D && \text{(10c)}\\
\boldsymbol{P} \cdot \boldsymbol{n} &= \boldsymbol{g}_N, && \text{on } \Gamma_N, && \text{(10d)}
\end{aligned}
$$

Here $\boldsymbol{\varphi}$ is the position vector, $\boldsymbol{F}$ the deformation gradient, and $\boldsymbol{P}$ the first Piola-Kirchhoff tensor.

Note that

$$
\boldsymbol{\varphi} = \boldsymbol{u} + \boldsymbol{x}, \qquad \boldsymbol{F} = \nabla \boldsymbol{u} + \boldsymbol{I}. \tag{11}
$$

The vector $\boldsymbol{\varphi}$ describes the deformation of the elastic body Ω.

Hyperelastic materials

For hyperelastic materials, the first Piola-Kirchhoff tensor is defined by

$$\boldsymbol{P} = \frac{\partial \psi}{\partial \boldsymbol{F}}, \tag{12}$$

where ψ is a strain energy density function.

Neo-Hookean model:

$$\psi = \frac{\mu}{2}\left(\mathrm{trace}(\boldsymbol{F}^T\boldsymbol{F}) - 3 - 2\ln(\det(\boldsymbol{F}))\right) + \frac{\lambda}{2}\left(\ln(\det(\boldsymbol{F}))\right)^2 \tag{13}$$

where μ and λ are the Lamé parameters

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{14}$$

Here E is Young's modulus and ν is Poisson's ratio.

HDG Method for Nonlinear Elasticity

We seek $(\boldsymbol{F}_h, \boldsymbol{\varphi}_h, \widehat{\boldsymbol{\varphi}}_h) \in \boldsymbol{Q}_h^k \times \boldsymbol{V}_h^k \times \boldsymbol{M}_h^k$ such that

$$
\begin{aligned}
(\boldsymbol{F}_h, \boldsymbol{E})_{\mathcal{T}_h} + (\boldsymbol{\varphi}_h, \nabla \cdot \boldsymbol{E})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\varphi}}_h, \boldsymbol{E} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\
(\boldsymbol{P}(\boldsymbol{F}_h), \nabla \boldsymbol{w})_{\mathcal{T}_h} - \left\langle \widehat{\boldsymbol{P}}_h \boldsymbol{n}, \boldsymbol{w} \right\rangle_{\partial \mathcal{T}_h} &= (\boldsymbol{b}, \boldsymbol{w})_{\mathcal{T}_h}, \\
\left\langle \widehat{\boldsymbol{P}}_h \boldsymbol{n}, \boldsymbol{\mu} \right\rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} + \langle \widehat{\boldsymbol{\varphi}}_h - \boldsymbol{g}_D, \boldsymbol{\mu} \rangle_{\Gamma_D} + \left\langle \widehat{\boldsymbol{P}}_h \boldsymbol{n} - \boldsymbol{g}_N, \boldsymbol{\mu} \right\rangle_{\Gamma_N} &= 0,
\end{aligned}
\tag{15}
$$

for all $(\boldsymbol{E}, \boldsymbol{w}, \boldsymbol{\mu}) \in \boldsymbol{Q}_h^k \times \boldsymbol{V}_h^k \times \boldsymbol{M}_h^k$, where

$$
\widehat{\boldsymbol{P}}_h \boldsymbol{n} = \boldsymbol{P}(\boldsymbol{F}_h) \cdot \boldsymbol{n} - \boldsymbol{\tau}(\boldsymbol{\varphi}_h - \widehat{\boldsymbol{\varphi}}_h). \tag{16}
$$

Here $\boldsymbol{\tau}$ is the stabilization tensor.

Linear Elastodynamics

The governing equations:

$$\begin{aligned} 2\boldsymbol{\mathcal{S}}\boldsymbol{\sigma} - (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) &= 0, && \text{in } \Omega, \\ \ddot{\boldsymbol{u}} - \nabla \cdot \boldsymbol{\sigma} &= \boldsymbol{b}, && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{g}_D, && \text{on } \Gamma_D, \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} &= \boldsymbol{g}_N, && \text{on } \Gamma_N. \end{aligned} \tag{17}$$

Introducing $\boldsymbol{v} = \dot{\boldsymbol{u}}$, we obtain

$$\begin{aligned} 2\boldsymbol{\mathcal{S}}\dot{\boldsymbol{\sigma}} - (\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T) &= 0, && \text{in } \Omega, \\ \dot{\boldsymbol{v}} - \nabla \cdot \boldsymbol{\sigma} &= \boldsymbol{b}, && \text{in } \Omega, \\ \boldsymbol{v} &= \dot{\boldsymbol{g}}_D, && \text{on } \Gamma_D, \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} &= \boldsymbol{g}_N, && \text{on } \Gamma_N. \end{aligned} \tag{18}$$

HDG Method for Linear Elastodynamics

We seek $(\boldsymbol{\sigma}_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \in \boldsymbol{S}_h^k \times \boldsymbol{V}_h^k \times \boldsymbol{M}_h^k$ such that

$$
\begin{aligned}
(\boldsymbol{\mathcal{S}}\dot{\boldsymbol{\sigma}}_h, \boldsymbol{\zeta})_{\mathcal{T}_h} + (\boldsymbol{v}_h, \nabla \cdot \boldsymbol{\zeta})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{v}}_h, \boldsymbol{\zeta} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\
(\dot{\boldsymbol{v}}_h, \boldsymbol{w})_{\mathcal{T}_h} + (\boldsymbol{\sigma}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}, \boldsymbol{w} \rangle_{\partial \mathcal{T}_h} &= (\boldsymbol{b}, \boldsymbol{w})_{\mathcal{T}_h}, \\
\langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \partial\Omega} + \langle \widehat{\boldsymbol{v}}_h - \dot{\boldsymbol{g}}_D, \boldsymbol{\mu} \rangle_{\Gamma_D} + \langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n} - \boldsymbol{g}_N, \boldsymbol{\mu} \rangle_{\Gamma_N} &= 0,
\end{aligned}
\tag{19}
$$

for all $(\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\mu}) \in \boldsymbol{S}_h^k \times \boldsymbol{V}_h^k \times \boldsymbol{M}_h^k$, where

$$
\widehat{\boldsymbol{\sigma}}_h \boldsymbol{n} = \boldsymbol{\sigma}_h \cdot \boldsymbol{n} - \boldsymbol{\tau}(\boldsymbol{v}_h - \widehat{\boldsymbol{v}}_h). \tag{20}
$$

Here $\boldsymbol{\tau}$ is the stabilization tensor.

Nonlinear Elastodynamics

We consider the dynamics for a nonlinear elastic body

$$
\begin{aligned}
\boldsymbol{F} - \nabla \boldsymbol{\varphi} &= 0 && \text{in } \Omega, && \text{(21a)}\\
\ddot{\boldsymbol{\varphi}} - \nabla \cdot \boldsymbol{P}(\boldsymbol{F}) &= \boldsymbol{b} && \text{in } \Omega && \text{(21b)}\\
\boldsymbol{\varphi} &= \boldsymbol{g}_D, && \text{on } \Gamma_D && \text{(21c)}\\
\boldsymbol{P} \cdot \boldsymbol{n} &= \boldsymbol{g}_N, && \text{on } \Gamma_N, && \text{(21d)}
\end{aligned}
$$

Introducing $\boldsymbol{v} = \dot{\boldsymbol{\varphi}}$, we obtain

$$
\begin{aligned}
\dot{\boldsymbol{F}} - \nabla \boldsymbol{v} &= 0 && \text{in } \Omega,\\
\dot{\boldsymbol{v}} - \nabla \cdot \boldsymbol{P}(\boldsymbol{F}) &= \boldsymbol{b} && \text{in } \Omega\\
\boldsymbol{v} &= \dot{\boldsymbol{g}}_D, && \text{on } \Gamma_D\\
\boldsymbol{P} \cdot \boldsymbol{n} &= \boldsymbol{g}_N, && \text{on } \Gamma_N,
\end{aligned}
\qquad (22)
$$

HDG Method for Nonlinear Elastodynamics

We seek $(\boldsymbol{F}_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \in \boldsymbol{Q}_h^k \times \boldsymbol{V}_h^k \times \boldsymbol{M}_h^k$ such that

$$
\begin{aligned}
(\dot{\boldsymbol{F}}_h, \boldsymbol{E})_{\mathcal{T}_h} + (\boldsymbol{v}_h, \nabla \cdot \boldsymbol{E})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{v}}_h, \boldsymbol{E} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\
(\dot{\boldsymbol{v}}_h, \boldsymbol{w})_{\mathcal{T}_h} + (\boldsymbol{P}(\boldsymbol{F}_h), \nabla \boldsymbol{w})_{\mathcal{T}_h} - \left\langle \widehat{\boldsymbol{P}}_h \boldsymbol{n}, \boldsymbol{w} \right\rangle_{\partial \mathcal{T}_h} &= (\boldsymbol{b}, \boldsymbol{w})_{\mathcal{T}_h}, \\
\left\langle \widehat{\boldsymbol{P}}_h \boldsymbol{n}, \boldsymbol{\mu} \right\rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} + \langle \widehat{\boldsymbol{v}}_h - \dot{\boldsymbol{g}}_D, \boldsymbol{\mu} \rangle_{\Gamma_D} + \left\langle \widehat{\boldsymbol{P}}_h \boldsymbol{n} - \boldsymbol{g}_N, \boldsymbol{\mu} \right\rangle_{\Gamma_N} &= 0,
\end{aligned}
\tag{23}
$$

for all $(\boldsymbol{E}, \boldsymbol{w}, \boldsymbol{\mu}) \in \boldsymbol{Q}_h^k \times \boldsymbol{V}_h^k \times \boldsymbol{M}_h^k$, where

$$
\widehat{\boldsymbol{P}}_h \boldsymbol{n} = \boldsymbol{P}(\boldsymbol{F}_h) \cdot \boldsymbol{n} - \boldsymbol{\tau}(\boldsymbol{v}_h - \widehat{\boldsymbol{v}}_h). \tag{24}
$$

Here $\boldsymbol{\tau}$ is the stabilization tensor.

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