HDG Methods for Incompressible Flow

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Outline

Goals:

- Stokes flow
- Incompressible Navier-Stokes flow

Notation: Div, Curl, Grad

Divergence (**Div**) of a vector:

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

Rotation (**Curl**) of a vector:

$$\nabla \times \boldsymbol{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \; \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \; \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

Gradient (**Grad**) of a vector:

$$\boldsymbol{L} \equiv \nabla \boldsymbol{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

Notation: Stress tensors

Total stress tensor:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\tau}$$

where p is the pressure and $\boldsymbol{\tau}$ is the viscous stress tensor.

For linear isotropic (Newtonian) medium, $\boldsymbol{\tau}$ is given by

$$\boldsymbol{\tau} = \lambda(\nabla \cdot \boldsymbol{u})\boldsymbol{I} + \nu(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T).$$

where λ is the bulk viscosity and ν is the dynamic viscosity.

For incompressible flow $\nabla \cdot \boldsymbol{u} = 0$, the stress tensor is

$$\boldsymbol{\tau} = \nu(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T).$$

Stokes Equations: Stress Formulation

The Stokes equations are

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma} &= \boldsymbol{s}, && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0, && \text{in } \Omega. \end{aligned} \tag{1}$$

Here $\boldsymbol{s}$ is a forcing term.

Since $\nabla \cdot \boldsymbol{u} = 0$, we obtain

$$\begin{aligned} \boldsymbol{\tau} - \nu(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) &= 0, && \text{in } \Omega, \\ \nabla \cdot (p\boldsymbol{I} - \boldsymbol{\tau}) &= \boldsymbol{s}, && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0, && \text{in } \Omega. \end{aligned} \tag{2}$$

This is the **stress formulation** of the Stokes flow.

Stokes Equations: Gradient Formulation

Furthermore, we note that

$$\nabla \cdot (\nu(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)) = \nabla \cdot (\nu \nabla \boldsymbol{u}) + \nabla \nu (\nabla \cdot \boldsymbol{u}) = \nabla \cdot (\nu \nabla \boldsymbol{u}) \quad (3)$$

We thus have

$$\begin{aligned} \boldsymbol{L} - \nabla \boldsymbol{u} &= 0, && \text{in } \Omega, \\ \nabla \cdot (p\boldsymbol{I} - \nu \boldsymbol{L}) &= \boldsymbol{s}, && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0, && \text{in } \Omega. \end{aligned} \quad (4)$$

This is the **gradient formulation** of the Stokes flow.

Stokes Equations: Boundary Condition

For simplicity we consider the boundary condition:

$$\boldsymbol{u} = \boldsymbol{g}, \quad \text{on } \partial\Omega. \tag{5}$$

To ensure wellposedness of the problem, we impose the average pressure condition:

$$\int_{\Omega} p = 0. \tag{6}$$

In addition, the boundary data $\boldsymbol{g}$ must satisfy a compatibility condition:

$$\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} = 0. \tag{7}$$

Approximation Spaces

We introduce the following spaces

$$W_h^k = \{w \in L^2(\mathcal{T}_h) \; : \; w|_K \in \mathcal{P}_k(K), \forall K \in \mathcal{T}_h\},$$

$$\boldsymbol{V}_h^k = \{\boldsymbol{v} \in [L^2(\mathcal{T}_h)]^d \; : \; \boldsymbol{v}|_K \in [\mathcal{P}_k(K)]^d, \forall K \in \mathcal{T}_h\},$$

$$\boldsymbol{Q}_h^k = \{\boldsymbol{E} \in [L^2(\mathcal{T}_h)]^{d\times d} \; : \; \boldsymbol{E}|_K \in [\mathcal{P}_k(K)]^{d\times d}, \forall K \in \mathcal{T}_h\},$$

$$\boldsymbol{M}_h^k = \{\boldsymbol{\mu} \in [L^2(\mathcal{E}_h)]^d \; : \; \boldsymbol{\mu}|_F \in [\mathcal{P}_k(F)]^d, \forall F \in \mathcal{E}_h\}.$$

Inner Products

We define the volume inner products as

$$(w, v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (w, v)_K, \qquad (\boldsymbol{w}, \boldsymbol{v})_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{d} (w_i, v_i)_K, \tag{8}$$

$$(\boldsymbol{W}, \boldsymbol{V})_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{d} \sum_{j=1}^{d} (W_{ij}, V_{ij})_K, \tag{9}$$

and the boundary inner product as

$$\langle w, v \rangle_{\partial \mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \langle w, v \rangle_{\partial K}, \quad \langle \boldsymbol{w}, \boldsymbol{v} \rangle_{\partial \mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{d} \langle w_i, v_i \rangle_{\partial K}, \tag{10}$$

where

$$(w, v)_K = \int_K wv, \qquad \langle w, v \rangle_{\partial K} = \int_{\partial K} wv. \tag{11}$$

We seek $(\boldsymbol{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h) \in \boldsymbol{Q}_h^k \times \boldsymbol{V}_h^k \times W_h^k \times \boldsymbol{M}_h^k$ such that

$$
\begin{aligned}
(\boldsymbol{L}_h, \boldsymbol{E})_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla \cdot \boldsymbol{E})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{E} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\
(\nu \boldsymbol{L}_h - p_h \boldsymbol{I}, \nabla \boldsymbol{w})_{\mathcal{T}_h} + \left\langle (-\nu \widehat{\boldsymbol{L}}_h + \widehat{p}_h \boldsymbol{I}) \boldsymbol{n}, \boldsymbol{w} \right\rangle_{\partial \mathcal{T}_h} &= (\boldsymbol{s}, \boldsymbol{w})_{\mathcal{T}_h}, \\
-(\boldsymbol{u}_h, \nabla q)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, q \rangle_{\partial \mathcal{T}_h} &= 0, \\
\left\langle (-\nu \widehat{\boldsymbol{L}}_h + \widehat{p}_h \boldsymbol{I}) \boldsymbol{n}, \boldsymbol{\mu} \right\rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} + \langle \widehat{\boldsymbol{u}}_h - \boldsymbol{g}, \boldsymbol{\mu} \rangle_{\partial \Omega} &= 0, \\
(p_h, 1)_{\mathcal{T}_h} &= 0,
\end{aligned}
\tag{12}
$$

for all $(\boldsymbol{E}, \boldsymbol{w}, q, \boldsymbol{\mu}) \in \boldsymbol{Q}_h^k \times \boldsymbol{V}_h^k \times W_h^k \times \boldsymbol{M}_h^k$, where

$$
(-\nu \widehat{\boldsymbol{L}}_h + \widehat{p}_h \boldsymbol{I}) \boldsymbol{n} = (-\nu \boldsymbol{L}_h + p_h \boldsymbol{I}) \cdot \boldsymbol{n} + \boldsymbol{S}(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h). \tag{13}
$$

Here $\boldsymbol{S}$ is the stabilization tensor.

Ill-posedness of the Local Problem

Let us consider the local problem for the HDG method: Find $(\boldsymbol{L}_h, \boldsymbol{u}_h, p_h) \in \boldsymbol{Q}_h^k \times \boldsymbol{V}_h^k \times W_h^k$ such that

$$
\begin{aligned}
(\boldsymbol{L}_h, \boldsymbol{E})_K + (\boldsymbol{u}_h, \nabla \cdot \boldsymbol{E})_K - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{E} \cdot \boldsymbol{n} \rangle_{\partial K} &= 0, \\
-(\nabla \cdot (\nu \boldsymbol{L}_h - p_h \boldsymbol{I}), \boldsymbol{w})_K + \langle \boldsymbol{S}(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \boldsymbol{w} \rangle_{\partial K} &= (\boldsymbol{s}, \boldsymbol{w})_K, \\
-(\boldsymbol{u}_h, \nabla q)_K + \langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, q \rangle_{\partial K} &= 0.
\end{aligned}
\tag{14}
$$

Note that if p_h is a solution, then $p_h + C$ is also a solution of the problem. This implies that the local problem is ill-posed. Hence, it is not solvable at the element level.

There are two different approaches to address this issue.

Mean of The Pressure: The Local Problem

In the first approach, we introduce $\bar{\varrho}_h \in W_h^0$ and redefine the local problem as: Find $(\boldsymbol{L}_h, \boldsymbol{u}_h, p_h) \in \boldsymbol{Q}_h^k \times \boldsymbol{V}_h^k \times W_h^k$ such that

$$
\begin{aligned}
(\boldsymbol{L}_h, \boldsymbol{E})_K + (\boldsymbol{u}_h, \nabla \cdot \boldsymbol{E})_K - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{E} \cdot \boldsymbol{n} \rangle_{\partial K} &= 0, \\
-(\nabla \cdot (\nu \boldsymbol{L}_h - p_h \boldsymbol{I}), \boldsymbol{w})_K + \langle \boldsymbol{S}(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \boldsymbol{w} \rangle_{\partial K} &= (\boldsymbol{s}, \boldsymbol{w})_K, \\
-(\boldsymbol{u}_h, \nabla q)_K + \langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, q \rangle_{\partial K} &= 0, \\
\frac{1}{|K|} \int_K p_h - \bar{\varrho}_h|_K &= 0,
\end{aligned}
\tag{15}
$$

for all $(\boldsymbol{E}, \boldsymbol{w}, q) \in [\mathcal{P}_k(K)]^{d \times d} \times [\mathcal{P}_k(K)]^d \times \mathcal{P}_k(K)$.

The new unknown $\bar{\varrho}_h$ is the mean of the approximate pressure.

Mean of The Pressure: The Global Problem

Find $(\widehat{\boldsymbol{u}}_h, \bar{\varrho}_h) \in \boldsymbol{M}_h^k \times W_h^0$ such that

$$\begin{aligned}
\langle (-\nu \boldsymbol{L}_h + p_h \boldsymbol{I})\boldsymbol{n} + \boldsymbol{S}(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} + \langle \widehat{\boldsymbol{u}}_h - \boldsymbol{g}, \boldsymbol{\mu} \rangle_{\partial \Omega} &= 0, \\
\langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, \eta \rangle_{\partial \mathcal{T}_h} &= 0, \\
(\varrho_h, 1)_{\mathcal{T}_h} &= 0,
\end{aligned} \tag{16}$$

for all $(\boldsymbol{\mu}, \eta) \in \boldsymbol{M}_h^k \times W_h^0$.
The associated global matrix system is given by

$$\begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B}^T & 0 \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{u}} \\ \bar{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbb{F} \\ 0 \end{bmatrix} \tag{17}$$

where $(\widehat{\mathbf{u}}, \bar{\mathbf{p}})$ represents the vectors of DOFs of $(\widehat{\boldsymbol{u}}_h, \bar{\varrho}_h)$.

Augmented Lagrangian: The Local Problem

In this approach, we introduce an artificial time derivative to the local problem as: Find $(\boldsymbol{L}_h, \boldsymbol{u}_h, p_h) \in \boldsymbol{Q}_h^k \times \boldsymbol{V}_h^k \times W_h^k$ such that

$$
\begin{aligned}
(\boldsymbol{L}_h, \boldsymbol{E})_K + (\boldsymbol{u}_h, \nabla \cdot \boldsymbol{E})_K - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{E} \cdot \boldsymbol{n} \rangle_{\partial K} &= 0, \\
-(\nabla \cdot (\nu \boldsymbol{L}_h - p_h \boldsymbol{I}), \boldsymbol{w})_K + \langle \boldsymbol{S}(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \boldsymbol{w} \rangle_{\partial K} &= (\boldsymbol{s}, \boldsymbol{w})_K, \\
\left(\frac{\partial p_h}{\partial t}, q \right)_K - (\boldsymbol{u}_h, \nabla q)_K + \langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, q \rangle_{\partial K} &= 0
\end{aligned}
\tag{18}
$$

for all $(\boldsymbol{E}, \boldsymbol{w}, q) \in [\mathcal{P}_k(K)]^{d \times d} \times [\mathcal{P}_k(K)]^d \times \mathcal{P}_k(K)$ and $t \in (0, \infty)$.

We then integrate the above system in time using the BE scheme to obtain the steady-state solution.

Augmented Lagrangian: The Local Problem

We find $(\boldsymbol{L}_h^n, \boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{Q}_h^k \times \boldsymbol{V}_h^k \times W_h^k$ such that

$$
\begin{aligned}
(\boldsymbol{L}_h^n, \boldsymbol{E})_K + (\boldsymbol{u}_h^n, \nabla \cdot \boldsymbol{E})_K - \langle \widehat{\boldsymbol{u}}_h^n, \boldsymbol{E} \cdot \boldsymbol{n} \rangle_{\partial K} &= 0, \\
-(\nabla \cdot (\nu \boldsymbol{L}_h^n - p_h^n \boldsymbol{I}), \boldsymbol{w})_K + \langle \boldsymbol{S}(\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n), \boldsymbol{w} \rangle_{\partial K} &= (\boldsymbol{s}, \boldsymbol{w})_K, \\
\left(\frac{p_h^n}{\Delta t}, q \right)_K - (\boldsymbol{u}_h^n, \nabla q)_K + \langle \widehat{\boldsymbol{u}}_h^n \cdot \boldsymbol{n}, q \rangle_{\partial K} &= \left(\frac{p_h^{n-1}}{\Delta t}, q \right)_K
\end{aligned}
\tag{19}
$$

for all $(\boldsymbol{E}, \boldsymbol{w}, q) \in [\mathcal{P}_k(K)]^{d \times d} \times [\mathcal{P}_k(K)]^d \times \mathcal{P}_k(K)$.

We then integrate the above system in time using the BE scheme to obtain the steady-state solution.

Augmented Lagrangian: The Global Problem

Find $\widehat{\boldsymbol{u}}_h^n$ such that

$$\langle(-\nu \boldsymbol{L}_h^n + p_h^n \boldsymbol{I})\boldsymbol{n} + \boldsymbol{S}(\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n), \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} + \langle\widehat{\boldsymbol{u}}_h^n - \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega} = 0, \tag{20}$$

for all $\boldsymbol{\mu} \in \boldsymbol{M}_h^k$.

Applying static condensation we obtain

$$\mathbb{A}\widehat{\mathbf{u}}^n = \mathbb{F}^n \tag{21}$$

where $\widehat{\mathbf{u}}^n$ represents the vectors of DOFs of $\widehat{\boldsymbol{u}}_h^n$.

Time marching the local-global system until

$$\frac{\|\widehat{\boldsymbol{u}}_h^n - \widehat{\boldsymbol{u}}_h^{n-1}\|_{\mathcal{T}_h}}{\|\widehat{\boldsymbol{u}}_h^n\|_{\mathcal{T}_h}} \leq \text{TOL}\ . \tag{22}$$

Summary

- The mean of the pressure approach yields a global system typical of the saddle-point problems.
- The Augmented Lagrangian approach yields a global system without the mean of the pressure. However, it requires a number of iterations (typically small) to converge.
- It is possible to obtain a new approximate velocity $\boldsymbol{u}_h^\star$ which is ***divergence-free*** and $\boldsymbol{H}$***(div)-conforming***. Moreover, $\boldsymbol{u}_h^\star$ converges with order $k+2$ for smooth problems. However, the local post processing is rather complicated.

Incompressible Navier-Stokes Equations

We consider the incompressible NS equations

$$
\begin{aligned}
-\nu \Delta \boldsymbol{u} + \nabla p + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) &= \boldsymbol{s}, && \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u} &= 0, && \text{in } \Omega,
\end{aligned}
\tag{23}
$$

with boundary conditions

$$
\begin{aligned}
\boldsymbol{u} = \boldsymbol{g}_D, &\quad \text{on } \partial\Omega_D, \\
\boldsymbol{B}\boldsymbol{n} = \boldsymbol{g}_N, &\quad \text{on } \partial\Omega_N,
\end{aligned}
\tag{24}
$$

Here the operator $\boldsymbol{B}$ is linear and depends on $(\boldsymbol{L}, \boldsymbol{u}, p)$.

We rewrite the incompressible NS equations as

$$
\begin{aligned}
\mathbf{L} - \nabla \boldsymbol{u} &= 0, && \text{in } \Omega \\
-\nabla \cdot (\nu \mathbf{L} - p\mathbf{I} - \boldsymbol{u} \otimes \boldsymbol{u}) &= \boldsymbol{s}, && \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u} &= 0, && \text{in } \Omega.
\end{aligned}
\tag{25}
$$

We seek $(\boldsymbol{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h) \in \boldsymbol{\mathcal{Q}}_h^k \times \boldsymbol{\mathcal{V}}_h^k \times \mathcal{W}_h^k \times \boldsymbol{\mathcal{M}}_h^k$ such that

$$
\begin{aligned}
(\boldsymbol{L}_h, \boldsymbol{E})_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla \cdot \boldsymbol{E})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{E} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\
(\nu \boldsymbol{L}_h - p_h \boldsymbol{I} - \boldsymbol{u}_h \otimes \boldsymbol{u}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{h}}_h, \boldsymbol{w} \right\rangle_{\partial \mathcal{T}_h} &= (\boldsymbol{s}, \boldsymbol{w})_{\mathcal{T}_h}, \\
-(\boldsymbol{u}_h, \nabla q)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, q \rangle_{\partial \mathcal{T}_h} &= 0, \\
\left\langle \widehat{\boldsymbol{h}}_h, \boldsymbol{\mu} \right\rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} + \langle \widehat{\boldsymbol{u}}_h - \boldsymbol{g}_D, \boldsymbol{\mu} \rangle_{\partial \Omega_D} + \left\langle \widehat{\boldsymbol{b}}_h, \boldsymbol{\mu} \right\rangle_{\partial \Omega_N} &= 0,
\end{aligned} \tag{26}
$$

for all $(\boldsymbol{E}, \boldsymbol{w}, q, \boldsymbol{\mu}) \in \boldsymbol{\mathcal{Q}}_h^k \times \boldsymbol{\mathcal{V}}_h^k \times \mathcal{W}_h^k \times \boldsymbol{\mathcal{M}}_h^k$, where

$$
\widehat{\boldsymbol{h}}_h = (-\nu \boldsymbol{L}_h + p_h \boldsymbol{I} + \widehat{\boldsymbol{u}}_h \otimes \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n} + \boldsymbol{S}(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h). \tag{27}
$$

Here $\boldsymbol{S}$ is the stabilization tensor.

Incompressible NS Equations: Stabilization Tensor

The stabilization tensor $\boldsymbol{S}$ can be defined as

$$\boldsymbol{S} = \tau \boldsymbol{I}. \tag{28}$$

Here the parameter τ should be chosen such that

$$\tau \geq |\boldsymbol{u} \cdot \boldsymbol{n}| + \frac{\nu}{\ell}. \tag{29}$$

Incompressible NS Equations: Boundary Conditions

Condition Type	$\boldsymbol{B}$	$\widehat{\boldsymbol{b}}_h$
stress	$-\nu(\boldsymbol{L}+\boldsymbol{L}^T)+p\boldsymbol{I}$	$(-\nu(\boldsymbol{L}_h+\boldsymbol{L}_h^T)+p_h\boldsymbol{I})\boldsymbol{n}+\boldsymbol{S}(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h)$
viscous stress*	$-\nu(\boldsymbol{L}+\boldsymbol{L}^T)$	$-\nu(\boldsymbol{L}_h+\boldsymbol{L}_h^T)\boldsymbol{n}+\boldsymbol{S}(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h)$
vorticity+pressure	$-\nu(\boldsymbol{L}-\boldsymbol{L}^T)+p\boldsymbol{I}$	$(-\nu(\boldsymbol{L}_h-\boldsymbol{L}_h^T)+p_h\boldsymbol{I})\boldsymbol{n}+\boldsymbol{S}(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h)$
vorticity*,†	$-\nu(\boldsymbol{L}-\boldsymbol{L}^T)$	$-\nu(\boldsymbol{L}_h-\boldsymbol{L}_h^T)\boldsymbol{n}+\boldsymbol{S}(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h)$
gradient+pressure	$-\nu\boldsymbol{L}+p\boldsymbol{I}$	$(-\nu\boldsymbol{L}_h+p_h\boldsymbol{I})\boldsymbol{n}+\boldsymbol{S}(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h)$
gradient*	$-\nu\boldsymbol{L}$	$-\nu\boldsymbol{L}_h\cdot\boldsymbol{n}+\boldsymbol{S}(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h)$

Table: Neumman boundary conditions for incompressible flow. Note that the asterisk symbol * indicates that $(p_h, 1)_\Omega = 0$ is needed. The dagger symbol $\dagger$ indicates that a Dirichlet boundary condition for the normal component of the velocity is needed on $\partial\Omega_N$.

Incompressible NS Equations: Implementation Steps

- Use the Raphson-Newton to obtain the linearized problem
- Apply either the mean of the pressure approach or the Augmented Lagrangian approach to the linearized problem
- Solve the linear system
- Update the solution
- Repeat the process until convergence.

Local Postprocessing

Find $\boldsymbol{u}_h^\star \in [\mathcal{P}_{k+1}(K)]^d$ such that

$$\begin{aligned}
\langle(\boldsymbol{u}_h^\star - \widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}, \mu\rangle_F &= 0, \quad \forall\, \mu \in \mathcal{P}_k(F), \forall F \in K, \\
\langle(\boldsymbol{n}\times\nabla)(\boldsymbol{u}_h^\star\cdot\boldsymbol{n}) & \\
-\boldsymbol{n}\times(\,\{\!\{\mathbf{L}_h^T\}\!\}\boldsymbol{n}), (\boldsymbol{n}\times\nabla)\mu\rangle_F &= 0, \quad \forall \mu \in \mathcal{P}_{k+1}(F)^\perp, \forall F \in K, \\
(\boldsymbol{u}_h^\star - \boldsymbol{u}_h, \nabla w)_K &= 0, \quad \forall\, w \in \mathcal{P}_k(K), \\
(\nabla\times\boldsymbol{u}_h^\star - \mathbf{L}_{21\,h} + \mathbf{L}_{12\,h}, w)_K &= 0, \quad \forall\, w \in \mathcal{P}_{k-1}(K),
\end{aligned}$$

where

$$\mathcal{P}_k(F)^\perp := \{w \in \mathcal{P}_k(F) :\ \langle w, \zeta\rangle_F = 0,\ \forall\, \zeta \in \mathcal{P}_{k-1}(F)\}.$$

Note that $\boldsymbol{u}_h^\star$ is ***incompressible*** and $\boldsymbol{H}$***(div)-conforming***.
Moreover, $\boldsymbol{u}_h^\star$ converges with order $k+2$ for smooth problems.

Incompressible NS Equations: Kovasznay Example

We consider the incompressible Navier-Stokes flow in $\Omega = (-0.5, 1.5) \times (0, 2)$ with the exact solution:

$$
\begin{aligned}
u_1 &= 1 - \exp(\lambda x_1)\cos(2\pi x_2), \\
u_2 &= \frac{\lambda}{2\pi}\exp(\lambda x_1)\sin(2\pi x_2), \\
p &= \frac{1}{2}\exp(2\lambda x_1),
\end{aligned}
$$

where $\lambda = \frac{Re}{2} - \sqrt{\frac{Re^2}{4} + 4\pi^2}$ and $Re = \frac{1}{\nu} = 10$ is the Reynolds number.

Incompressible NS Equations: Kovasznay Example

		HDG				Taylor-Hood FEM			
		$\|p - p_h\|_{\mathcal{T}_h}$		$\|\boldsymbol{u} - \boldsymbol{u}_h^\star\|_{\mathcal{T}_h}$		$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathcal{T}_h}$		$\|p - p_h\|_{\mathcal{T}_h}$	
k	$1/h$	error	order	error	order	error	order	error	order
	4	1.48e-1	−−	1.38e-1	−−	1.31e-1	−−	8.71e-2	−−
	8	9.02e-3	4.03	8.28e-3	4.05	1.34e-2	3.29	1.47e-2	2.56
2	16	9.32e-4	3.27	5.47e-4	3.92	1.52e-3	3.15	2.83e-3	2.38
	32	1.12e-4	3.06	3.75e-5	3.87	1.82e-4	3.06	6.56e-4	2.11
	64	1.38e-5	3.02	2.46e-6	3.93	2.25e-5	3.02	1.62e-4	2.02
	4	1.57e-2	−−	1.42e-2	−−	1.62e-2	−−	2.13e-2	−−
	8	7.93e-4	4.31	5.68e-4	4.64	1.20e-3	3.76	1.30e-3	4.03
3	16	5.01e-5	3.98	1.89e-5	4.91	7.78e-5	3.94	1.30e-4	3.32
	32	3.18e-6	3.98	6.37e-7	4.89	5.00e-6	3.96	1.54e-5	3.08
	64	2.00e-7	3.99	2.07e-8	4.94	3.15e-7	3.99	1.89e-6	3.02

Table: Comparison of the convergence of the L^2 errors in the pressure and velocity between the HDG method and the continuous Taylor-Hood FE method .

Incompressible NS Equations: Cylinder Flow

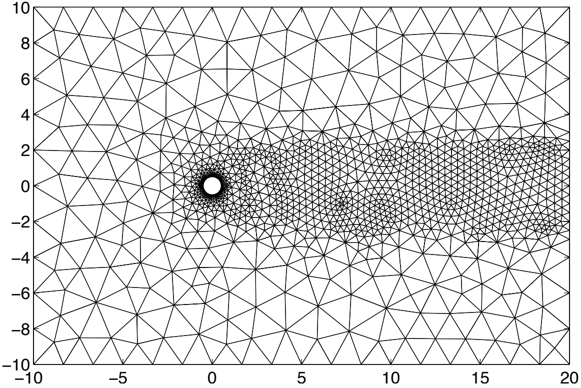

Figure: Finite element mesh.

Use $k = 4$ for spatial discretization and the DIRK(3,3) scheme with $\Delta t = 0.2$ for temporal discretization.

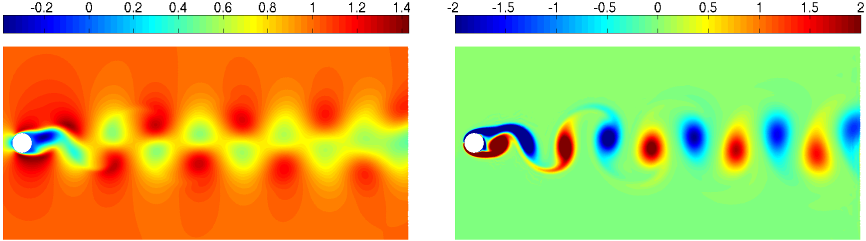

Figure: Horizontal velocity and vorticity at $t = 100$ for incompressible viscous flow past a circular cylinder at $Re = 200$.

The results agree well with previous calculations in the literature as well as with experimental measurements.

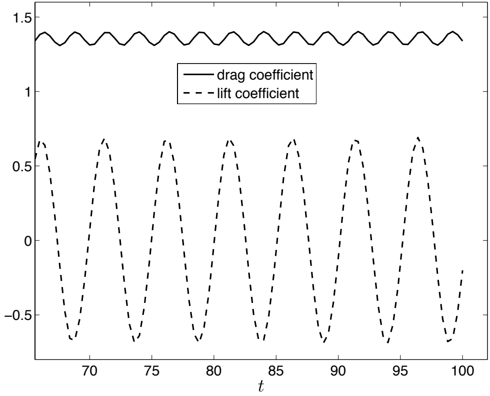

Figure: Time history of lift and drag coefficients for incompressible flow past a circular cylinder at $Re = 200$. The Strouhal number is 0.2.

References

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