HDG Methods for Maxwell's Equations

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Outline

Goals:

- Time-domain Maxwell's equations
- Frequency-domain Maxwell's equations

Notation: Div, Curl, Grad

Divergence (**Div**) of a vector:

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

Rotation (**Curl**) of a vector:

$$\nabla \times \boldsymbol{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

It is important to note that

$$\nabla \cdot (\nabla \times \boldsymbol{u}) = 0.$$

Maxwell's Equations

The Maxwell's equations are Faraday's law and Ampere's law for electromagnetics:

$$\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t}, \qquad \nabla \times \boldsymbol{H} = \boldsymbol{J} + \epsilon \frac{\partial \boldsymbol{E}}{\partial t} \tag{1}$$

together with Gauss's laws for electric and magnetic fields

$$\nabla \cdot (\mu \boldsymbol{H}) = 0, \qquad \nabla \cdot (\epsilon \boldsymbol{E}) = \rho. \tag{2}$$

Here $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ the magnetic field, $\boldsymbol{J}$ the current density, and ρ the charge density. Furthermore, μ and ϵ are the permeability and permittivity, respectivily.

The charge density ρ is related to the current density $\boldsymbol{J}$ through the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0. \tag{3}$$

Derivation of Gauss's Laws

Taking the divergence of (1) yields

$$0 = -\nabla \cdot \left(\mu \frac{\partial \boldsymbol{H}}{\partial t} \right), \qquad 0 = \nabla \cdot \boldsymbol{J} + \nabla \cdot \left(\epsilon \frac{\partial \boldsymbol{E}}{\partial t} \right) \tag{4}$$

Combining (3) and (4), we obtain

$$\frac{\partial}{\partial t}(\nabla \cdot \mu \boldsymbol{H}) = 0, \qquad \frac{\partial}{\partial t}(\nabla \cdot \epsilon \boldsymbol{E}) = \frac{\partial \rho}{\partial t}. \tag{5}$$

This equation implies that

$$\nabla \cdot \mu \boldsymbol{H} = 0, \qquad \nabla \cdot \epsilon \boldsymbol{E} = \rho. \tag{6}$$

Hence, the Gauss's laws (2) can be derived from (1) and (3). As a result, many numerical methods are developed to solve the Maxwell's equations (1) without the divergence constraints (2).

We will describe HDG methods for solving the following PDE system:

$$
\begin{aligned}
\mu \frac{\partial \boldsymbol{H}}{\partial t} + \nabla \times \boldsymbol{E} &= \boldsymbol{0}, \qquad \text{in } \Omega, \\
-\epsilon \frac{\partial \boldsymbol{E}}{\partial t} + \nabla \times \boldsymbol{H} &= \boldsymbol{J}, \qquad \text{in } \Omega,
\end{aligned} \tag{7}
$$

together with boundary conditions

$$
\begin{aligned}
\boldsymbol{n} \times \boldsymbol{E} &= \boldsymbol{g}_E, \qquad \text{on } \Gamma_E, \\
\boldsymbol{n} \times \boldsymbol{H} &= \boldsymbol{g}_H, \qquad \text{on } \Gamma_H,
\end{aligned} \tag{8}
$$

and initial conditions

$$
\boldsymbol{E}(t=0) = \boldsymbol{E}_0, \qquad \boldsymbol{H}(t=0) = \boldsymbol{H}_0. \tag{9}
$$

Approximation Spaces

We introduce the following spaces

$$\boldsymbol{V}_h^k = \{\boldsymbol{v} \in [L^2(\mathcal{T}_h)]^d \ : \ \boldsymbol{v}|_K \in [\mathcal{P}_k(K)]^d, \forall K \in \mathcal{T}_h\},$$

$$\boldsymbol{M}_h^k = \{\boldsymbol{\eta} \in [L^2(\mathcal{E}_h)]^d \ : \ \boldsymbol{\eta}|_F \in [\mathcal{P}_k(F)]^d, \boldsymbol{\eta} \cdot \boldsymbol{n} = 0, \forall F \in \mathcal{E}_h\}.$$

Note that $\boldsymbol{M}_h^k$ consists of vector-valued functions whose normal component is zero on any face. Hence, a member of $\boldsymbol{M}_h^k$ can be characterized by two tangential vectors on the faces: if $\boldsymbol{t}_1^F$ and $\boldsymbol{t}_2^F$ denote independent tangent vectors on F, we can write the restriction of $\boldsymbol{\eta} \in \boldsymbol{M}_h^k$ on F as

$$\boldsymbol{\eta}|_F = \alpha_1^F \boldsymbol{t}_1^F + \alpha_2^F \boldsymbol{t}_2^F, \tag{10}$$

where both α_1^F and α_2^F are polynomials of degree at most k on F.

Computing the Tangential Vectors

Given a unit normal vector $\boldsymbol{n} = (n_1, n_2, n_3)$, without loss of generality, we assume that

$$|n_1| \geq |n_2|, \qquad |n_1| \geq |n_3|. \tag{11}$$

Then the two tangential vectors can be computed as

$$\boldsymbol{t}_1 = (-n_2/n_1, 1, 0), \qquad \boldsymbol{t}_2 = (-n_3/n_1, 0, 1). \tag{12}$$

Note that the two tangential vectors would be the same if we set $\boldsymbol{n} = -(n_1, n_2, n_3)$.

HDG for Maxwell's Equations in Time Domain

We seek $(\boldsymbol{H}_h, \boldsymbol{E}_h, \widehat{\boldsymbol{E}}_h) \in \boldsymbol{V}_h^k \times \boldsymbol{V}_h^k \times \boldsymbol{M}_h^k$ such that

$$
\begin{aligned}
(\mu \dot{\boldsymbol{H}}_h, \boldsymbol{\zeta})_{\mathcal{T}_h} + (\boldsymbol{E}_h, \nabla \times \boldsymbol{\zeta})_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{E}}_h, \boldsymbol{\zeta} \times \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h} &= 0, \\
-(\epsilon \dot{\boldsymbol{E}}_h, \boldsymbol{w})_{\mathcal{T}_h} + (\boldsymbol{H}_h, \nabla \times \boldsymbol{w})_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{H}}_h, \boldsymbol{w} \times \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h} &= (\boldsymbol{J}, \boldsymbol{w})_{\mathcal{T}_h}, \\
\left\langle \boldsymbol{n} \times \widehat{\boldsymbol{H}}_h, \boldsymbol{\eta} \right\rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} + \left\langle \boldsymbol{n} \times \widehat{\boldsymbol{E}}_h - \boldsymbol{g}_E, \boldsymbol{\eta} \right\rangle_{\Gamma_E} & \\
+ \left\langle \boldsymbol{n} \times \widehat{\boldsymbol{H}}_h - \boldsymbol{g}_H, \boldsymbol{\eta} \right\rangle_{\Gamma_H} &= 0,
\end{aligned}
\tag{13}
$$

for all $(\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\eta}) \in \boldsymbol{V}_h^k \times \boldsymbol{V}_h^k \times \boldsymbol{M}_h^k$, where

$$
\widehat{\boldsymbol{H}}_h = \boldsymbol{H}_h + \tau (\boldsymbol{E}_h - \widehat{\boldsymbol{E}}_h) \times \boldsymbol{n}. \tag{14}
$$

Here τ is the stabilization function.

Maxwell's Equations in Frequency Domain

Assume that the electromagnetic field is time-periodic with frequency ω. Then we have

$$\boldsymbol{E}(t) = \mathrm{Real}[\boldsymbol{E}(\omega)e^{i\omega t}], \quad \boldsymbol{H}(t) = \mathrm{Real}[\boldsymbol{H}(\omega)e^{i\omega t}], \tag{15}$$

where $\boldsymbol{E}(\omega)$ and $\boldsymbol{H}(\omega)$ are the solution of the following PDE system:

$$\begin{aligned} i\mu\omega\boldsymbol{H} + \nabla\times\boldsymbol{E} &= \boldsymbol{0}, && \text{in } \Omega, \\ -i\epsilon\omega\boldsymbol{E} + \nabla\times\boldsymbol{H} &= \boldsymbol{J}, && \text{in } \Omega, \end{aligned} \tag{16}$$

together with boundary conditions

$$\begin{aligned} \boldsymbol{n}\times\boldsymbol{E} &= \boldsymbol{g}_E, && \text{on } \Gamma_E, \\ \boldsymbol{n}\times\boldsymbol{H} &= \boldsymbol{g}_H, && \text{on } \Gamma_H. \end{aligned} \tag{17}$$

Approximation Spaces

We introduce the following spaces

$$\boldsymbol{V}_h^k = \{\boldsymbol{v} \in [L^2(\mathcal{T}_h)]^d \ : \ \boldsymbol{v}|_K \in [\mathcal{C}_k(K)]^d, \forall K \in \mathcal{T}_h\},$$
$$\boldsymbol{M}_h^k = \{\boldsymbol{\eta} \in [L^2(\mathcal{E}_h)]^d \ : \ \boldsymbol{\eta}|_F \in [\mathcal{C}_k(F)]^d, \boldsymbol{\eta} \cdot \boldsymbol{n} = 0, \forall F \in \mathcal{E}_h\}. \tag{18}$$

where $\mathcal{C}_k(K)$ is the space of complex polynomials of degree at most k on K. Hence, a function $v \in \mathcal{C}_k(K)$ can be written as

$$v = v_{\mathrm{Real}} + i v_{\mathrm{Imag}} \tag{19}$$

where $v_{\mathrm{Real}} \in \mathcal{P}_k(K)$ and $v_{\mathrm{Imag}} \in \mathcal{P}_k(K)$ are the real part and imaginary part, respectively.

We seek $(\boldsymbol{H}_h, \boldsymbol{E}_h, \widehat{\boldsymbol{E}}_h) \in \boldsymbol{V}_h^k \times \boldsymbol{V}_h^k \times \boldsymbol{M}_h^k$ such that

$$
\begin{aligned}
(i\mu\omega \boldsymbol{H}_h, \boldsymbol{\zeta})_{\mathcal{T}_h} + (\boldsymbol{E}_h, \nabla \times \boldsymbol{\zeta})_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{E}}_h, \boldsymbol{\zeta} \times \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h} &= 0, \\
-(i\epsilon\omega \boldsymbol{E}_h, \boldsymbol{w})_{\mathcal{T}_h} + (\boldsymbol{H}_h, \nabla \times \boldsymbol{w})_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{H}}_h, \boldsymbol{w} \times \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h} &= (\boldsymbol{J}, \boldsymbol{w})_{\mathcal{T}_h}, \\
\left\langle \boldsymbol{n} \times \widehat{\boldsymbol{H}}_h, \boldsymbol{\eta} \right\rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} + \left\langle \boldsymbol{n} \times \widehat{\boldsymbol{E}}_h - \boldsymbol{g}_E, \boldsymbol{\eta} \right\rangle_{\Gamma_E} & \\
+ \left\langle \boldsymbol{n} \times \widehat{\boldsymbol{H}}_h - \boldsymbol{g}_H, \boldsymbol{\eta} \right\rangle_{\Gamma_H} &= 0,
\end{aligned} \tag{20}
$$

for all $(\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\eta}) \in \boldsymbol{V}_h^k \times \boldsymbol{V}_h^k \times \boldsymbol{M}_h^k$, where

$$
\widehat{\boldsymbol{H}}_h = \boldsymbol{H}_h + \tau(\boldsymbol{E}_h - \widehat{\boldsymbol{E}}_h) \times \boldsymbol{n}. \tag{21}
$$

Here τ is the stabilization function.

References

[NPC11] *Hybridizable discontinuous Galerkin methods for the time-harmonic Maxwell's equations*, J. Comp. Phys., 230 (2011), pp. 7151–7175.

Li L., Lanteri S., Perrussel R. (2013) *A Hybridizable Discontinuous Galerkin Method for Solving 3D Time-Harmonic Maxwell's Equations*. In: Cangiani A., Davidchack R., Georgoulis E., Gorban A., Levesley J., Tretyakov M. (eds) Numerical Mathematics and Advanced Applications 2011. Springer, Berlin, Heidelberg